We need an index of notation.

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS III: THE DOUBLE TREE CONSTRUCTION

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ABSTRACT. This is the third in a series of papers studying the finite type invariants of various w-knotted objects and their relationship to the Kashiwara-Vergne problem and Drinfel'd associators. In this paper we present a topological solution to the Kashiwara-Vergne problem. In particular we recover via a topological argument the Alkeseev-Enriquez-Torossian [AET] formula for explicit solutions of the Kashiwara-Vergne equations in terms of associators.

We study a class of w-knotted objects: knottings of *2-dimensional foams* and various associated features in four-dimensioanl space. We use a topological construction which we name the double tree construction to show that every *expansion* (also known as *universal finite type invariant*) of parenthesized braids extends first to an expansion of knotted trivalent graphs (a well known result), and then extends uniquely to an expansion of the w-knotted objects mentioned above.

In algebraic language, an expansion for parenthesized braids is the same as a *Drinfel'd associator* Φ, and an expansion for w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result provides a topological framework for the result of [AET] that "there is a formula for V in terms of Φ", along with an independent topological proof that the said formula works — namely that the equations satisfied by V follow from the equations satisfied by Φ.

CONTENTS

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4-dimansional space (called w-tangled foams), and explained the connection to Drinfel'd associators in terms of a relationship between 3-dimensional and 4-dimensional topology.

Another topological interpretation for the KV problem in terms of the Goldman-Turaev Lie bialgabre later emerged in [AKKN1, AKKN2], and the papers [M] and [AN] contain constructions of Goldman-Turaev expansions from the Kontsevich integral and the Knizhnik-Zamolodchikov connection, respectively.

In this paper we present a topological construction for a homomorphic universal finite type invariant of w-tangled foams, thereby giving a new topological proof for the KV conjecture. This construction also leads to an explicit formula for KV-solutions in terms of Drinfel'd associators, which we prove agrees with the formula [AET, Theorem 4].

Finally, we mention that a *circuit algebra*, which provides the algebraic structure to w-foams, were identified as equivalent to the operadic structure of a *wheeled prop* in [DHR1]. The symmetry groups of Kashiwara-Vergne solutions, called the Kashiwara-Vergne groups, are shown to be automorphism groups of the w-foam circuit algebra and its associated graded arrow diagrams in [DHR2]. The relationship between the symmetries of Drinfel'd associators – the Grothendieck-Teichmuller groups – and the Kashiwara-Vergne groups is described in the topological context of w-foams in the forthcoming paper [DHaR]

1.2.1. *Topology.* We begin by describing a chain of maps from "parenthesized braids" to "(signed) knotted trivalent graphs" to "w-tangled foams":

$$\mathcal{K} := \{uPaB \xrightarrow{cl} sKTG \xrightarrow{a} \widetilde{wTF}\}.$$

Let us first briefly elaborate on each of these spaces and maps.

Parenthesized braids are braids whose ends are ordered along two lines, the "bottom" and the "top", along with parenthetizations of the endpoints on the bottom and on the top. Two examples are shown in Figure 1. Parentehesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and composition is given by stacking. In addition to stacking, there are several operations defined on parenthesized braids: strand addition, removal and doubling. A detailed introduction to parenthesized braids is in [BN1].

Trivalent graphs are oriented graphs with three edges meeting at each vertex and whose vertices are equipped with a cyclic orientation of the incident edges. A knotted trivalent graph (KTG) is a framed embedding of a trivalent graph into $\mathbb{R}^3$. KTGs are studied from a finite type invariant point of view in [BND1]. In this paper we use a version of KTGs that was introduced and studied in [WKO2, Section 4.6], namely trivalent tangles with one or two ends and with some extra combinatorial information: trivalent vertices are equipped with a marked "distinguished edge" and signs. We call this space *sKTG* (for signed

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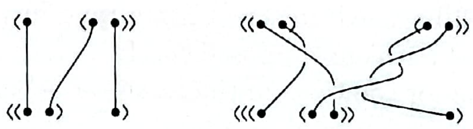
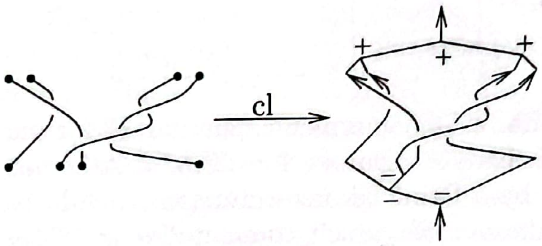


Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we omit the parentheses in parts of this paper.

KTGs), as in [WKO2]. An example is shown on the right. The space *sKTG* is also equipped with several operations: tangle insertion, sticking a 1-tangle onto an edge of another tangle, disjoint union of 1-tangles, edge unzip, and edge orientation switch (see [WKO2, Section 4.6] for details).

The space $\widetilde{wTF}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4], and will be introduced in detail in Section 2. It can be described as a circuit algebra (similar to a planar algebra but with non-planar connections allowed, see [WKO2, Section 2.4]) generated by certain features (various kinds of crossings and vertices, as well as "caps") modulo certain relations ("Reidemeister moves") and equipped with a number of auxiliary operations beyond the circuit algebra compositions. This Reidemeister theory conjecturally represents knotted tubes in 4-dimensional space with singular *foam vertices*, caps, and attached one-dimensional strings.

The map cl : *uPaB* $\to$ *sKTG* is the "closure map". Given a parenthesized braid, close up its top and bottom each by gluing a binary tree according to the parentetization; this produces a *sKTG* with the convention that all strands are oriented upwards, bottom vertices are negative, and top vertices are positive. An example is shown below.



(eq:cl)

The map a : *sKTG* $\to$ $\widetilde{wTF}$ arises combinatorially from the fact that all *sKTG* diagrams can be interpreted as elements of $\widetilde{wTF}$, and all *sKTG* Reidemeister moves are also imposed in $\widetilde{wTF}$. Topologically, it is an extended version of Satoh's tubing map, described in Remark 3.1.1 of [WKO2].

1.2.2. *Algebra.* The chain of maps $\mathcal{K}$ is an example of a general "algebraic structure", as discussed in [WKO2, Section 2.1]. An algebraic structure consists of a collection of objects belonging to a number of "spaces" or "different kinds", and operations that may be unary, binary, multinary or nullary, between these spaces. In this case there are many spaces (or kinds of objects): for example, parenthesized braids with specified bottom and top parenthetizations form one space, so do knottings of a given trivalent graph (skeleton). There is also a large collection of operations, consisting of all the internal operations of *uPaB*, *sKTG* and $\widetilde{wTF}$, as well as the maps a and cl.

In Sections 2.1 to 2.3 of [WKO2]we discuss associated graded structures and expansions for general algebraic structures. For any algebraic structure (think braids, or tangles with tangle composition), one allows formal linear compositions of elements of the same *kind* (think, same skeleton). Associated graded structures are taken with respect to the filtration by powers of the *augmentation ideal.* For the spaces *uPaB*, *sKTG* and $\widetilde{wTF}$, the associated graded spaces $\mathcal{A}^{hor}$, $\mathcal{A}^u$ and $\mathcal{A}^{sw}$ are the spaces of "horizontal chord diagrams on parenthesized strands", "chord diagrams on trivalent skeleta", and "arrow diagrams", as described in [BN1], [WKO2, Section 4.6], and Section 2 of this paper, respectively. As a result, the associated graded structure of $\mathcal{K}$ is

$$\mathcal{A} := \{\mathcal{A}^{hor} \xrightarrow{\text{cl}} \mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^{sw}\},$$

where cl and α are the maps induced by cl and a, respectively. More specifically, cl is the "closure of chord diagrams", and α is "replacing each chord with the sum of its two possible orientations", see [WKO2, Section 3.3].

An expansion [WKO2, Section 2.3] is a filtration-respecting map from an algebraic structure to its associated graded structure, whose associated graded map is the identity. In knot theory, expansions are also called universal finite type invariants. A homomorphic expansion is an expansion which behaves well with respect to the operations of the algebraic structure, that is, it intertwines each operation with its induced counterpart on the associated graded structure; for a detailed definition and introduction see [WKO2, Section 2.3]. Hence, a homomorphic expansion $Z : \mathcal{K} \to \mathcal{A}$ is a triple of homomorphic expansions Z^b, Z^u, and Z^w for $\mathcal{K}^b :=$ *uPaB*, $\mathcal{K}^u :=$ *sKTG* and $\mathcal{K}^w := \widetilde{wTF}$, respectively, so that the following diagram commutes:

$$\begin{tikzcd} \mathcal{K}: \arrow[d, "Z"] & \mathcal{K}^b \arrow[r, "\text{cl}"] \arrow[d, "Z^b"] & \mathcal{K}^u \arrow[r, "a"] \arrow[d, "Z^u"] & \mathcal{K}^w \arrow[d, "Z^w"] \\ \mathcal{A}: & \mathcal{A}^{hor} \arrow[r, "\text{cl}"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^w \end{tikzcd} \quad (1)$$

eq:MainDia

We recall (see [BN1]) that a homomorphic expansion Z^b for parenthesized braids is determined by a "horizontal chord associator" $\Phi = Z^b(|\diagup|)$. A homomorphic expansion Z^u of *sKTG* is also determined[2] by a Drinfel'd associator (horizontal chords or not; see [WKO2, Section 4.6]), so the significance of the left commutative square is to force the associator corresponding to Z^u to be a horizontal chord associator. In turn, Z^w is determined by a solution F (a close cousin of $V = Z^w(\Lsh)$) to the Kashiwara-Vergne problem (see [WKO2, Section 4.4 – 4.5]). The goal of this paper is to prove the following theorem, which, via the correspondence above, implies the KV conjecture:

thm:main **Theorem 1.1.** (1) *Assuming that $Z : \mathcal{K} \to \mathcal{A}$ exists, it is determined*[3] *by Z^u.*

(2) *There is a formula for V in terms of the Drinfel'd associator Φ associated to Z^u:*

what is C2.

$$V = C_1^{-1}C_2^{-1}C_{(12)}\varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2}\Phi(a_{23}, a_{43})\right), \quad (2)$$

eqn:AET

where a denotes a single arrow[4]*. This agrees*[5] *with the formula proven in* [AET].

(3) *Every Z^b extends to a Z.*

The key to the proof of the theorem is to show that the generator $\Lsh$ of $\widetilde{wTF}$ can be expressed in terms of the generator $|\diagup|$ of *uPaB* and the operations of $\mathcal{K}$. Assuming that Z exists, this yields a formula for V in terms of Φ.

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[2] With the exception of some minor normalization, see [WKO2, Section 4.6], in particular Lemma 4.14 and the paragraph following it.

[3] In fact, almost entirely determined by Z^b, with the exception of some minor normalization of Z^u which is not determined by an associator.

[4] The notation is explained in detail in Section 3.2

[5] Although the two formulas are written in different languages, and checking that they agree takes effort. See Section 3.2 and Appendix A.

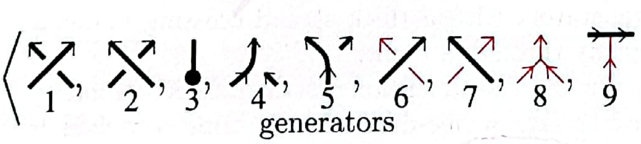
1.3. **Paper Structure.** In Section 2 we provide an overview of the space wTF^o of (oriented) w-foams and its extension with strings $\widetilde{\mathit{wTF}}$. We provide a brief review of definitions and crucial facts from [WKO2], and details of the extension. We prove that homomorphic expansions for wTF^o extend uniquely to homomorphic expansions for $\widetilde{\mathit{wTF}}$.

Section 3 makes up the bulk of the paper and is devoted to the proof of Theorem 1.1. In Section 3.1 we prove part (1). In Section 3.2 we deduce the formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators, proving part (2). In Section 3.3 we prove statement (3), the hardest part of the proof.

Section ?? is a short section of closing remarks, and in Appendix A we give an explicit comparison and equivalence between our formula in Part (2) and the ALekseev–Enriquez–Torossian [AET] formula.

2. The spaces $\widetilde{\mathit{wTF}}$ and $\mathcal{A}^{sw}$ in more detail

As mentioned in the introduction, $\widetilde{\mathit{wTF}}$ is a minor extension of the space wTF^o studied in [WKO2, Section 4.1 – 4.4]. It can be introduced as a planar algebra or as a circuit algebra; we will do the latter as it is simpler and more concise. Circuit algebras are defined in [WKO2, Section 2.4]; in short, they are similar to planar algebras but without the planarity requirement for "connecting strands". As in [WKO2], each generator and relation of $\widetilde{\mathit{wTF}}$ has a local topological interpretation. Recall from [WKO2, Sections 1.2, 3.4, 4.1] that wTF^o diagrams represent certain ribbon knotted tubes with foam vertices in $\mathbb{R}^4$, and the circuit algebra wTF^o is conjecturally a Reidemeister theory for this space (i.e., there is a surjection δ from the circuit algebra wTF^o to ribbon knotted tubes with foam vertices, and δ is conjectured to be an isomorphism). The space $\widetilde{\mathit{wTF}}$ extends wTF^o by adding one-dimensional strands to the picture. Note that in themselves, one dimensional strands in $\mathbb{R}^4$ are never knotted, however, they can be knotted *with* the two-dimensional tubes. In figures two-dimensional tubes will be denoted by thick lines and one dimensional strings by thin red lines. With this in mind, we define $\widetilde{\mathit{wTF}}$ as a circuit algebra defined in terms of generators and relations, and with some extra operations beyond circuit algebra compositions. Each generator, relation and operation has a local topological interpretation which provides much of the intuition behind the proofs. However, the corresponding Reidemeister theorem is only conjectural.

$\widetilde{\mathit{wTF}} = \mathrm{CA}\Big\langle$  | relations as in Section 2.2 | auxiliary operations as in Section 2.3 $\Big\rangle$

2.1. **The generators of $\widetilde{\mathit{wTF}}$.** We begin by discussing the local topological meaning of each generator shown above.

Figure 2. A string-tube vertex.

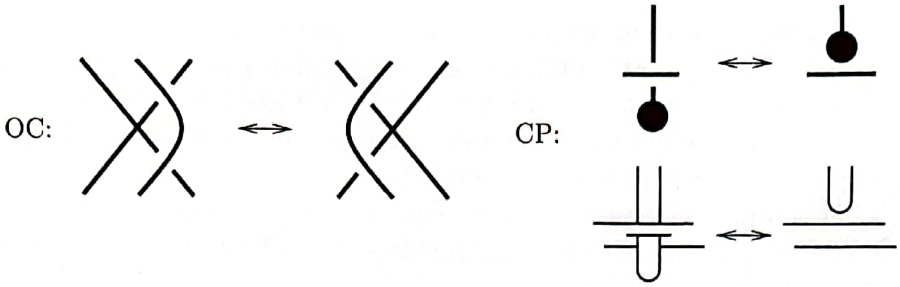
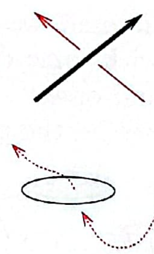


Figure 3. The OC and CP relations.

The first five generators are as described in [WKO2, Sections 4.1.1], we briefly recall their descriptions here. Knotted (more precisely, braided) tubes in $\mathbb{R}^4$ can equivalently be thought of as movies of flying rings in $\mathbb{R}^3$. The two crossings stand for movies where two rings trade places by the ring of the under strand flying through the ring of the over strand. The dotted end represents a tube "capped off" at the bottom by a disk. Generators 4 and 5 stand for singular "foam vertices", and will be referred to as the positive and negative vertex, respectively. The positive vertex represents the movie shown on the left: the right ring approaches the left ring from below, flies inside it and merges with it. The negative vertex represents a ring splitting and the inner ring flying out below and to the right. To be completely precise, $\widetilde{w\mathcal{T}F}$ as a circuit algebra has more vertex generators than shown above: the vertices appear with all possible orientations of the strands. However, all other versions can be obtained from the ones shown above using "orientation switch" operations (to be discussed in Section 2.3).



The thin red strands denote one dimensional strings in $\mathbb{R}^4$, or "flying points in $\mathbb{R}^3$". The crossings between the two types of strands (generators 6 and 7) represent "points flying through rings". For example, the picture on the left shows generator 6, where "the point on the right approaches the ring on the left from below, flies through the ring and out to the left above it". This explains why there are no generators with a thick strand crossing under a thin red strand: a ring cannot fly through a point.

Generator 9 is a trivalent vertex of 1-dimensional strings in $\mathbb{R}^4$. Finally, the last generator is a *mixed vertex:* a one-dimensional string attached to the wall of a 2-dimensional tube, as shown in Figure 2. All generators should be shown in all possible strand orientation combinations; we are suppressing this to save space.

where is it?

fig:wTFoRe

bsec:wrels

2.2. The relations. As a list, the relations for $\widetilde{w\mathcal{T}F}$ are the same as the relations for $w\mathcal{T}F^o$ [WKO2, Section 4.5]: {R1^s, R2, R3, R4, OC, CP}. Recall that R1^s is the weak (framed) version of the Reidemeister 1 move; R2 and R3 are the usual second and third Reidemeister moves; R4 allowes moving a strand over or under a vertex. OC stands for *Overcorssings Commute*, CP for *Cap Pullout*: these two relations are shown in Figure 3, for a detailed explanation see [WKO2, Section 4.1.2].

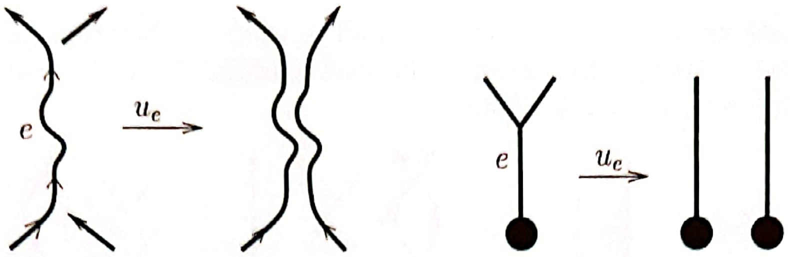


Figure 4. Unzip and disc unzip. fig:DiscUn

? p

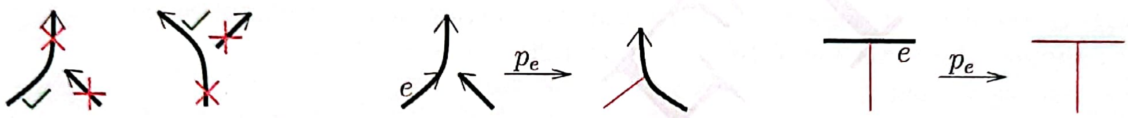


Figure 5. Puncture operations: the picture on the left shows which edges can be punctured at each vertex. The middle and right pictures show the effect of puncture operations. fig:punctu

map, and unzip is the act of "pushing the tube off of itself slightly in the framing direction". Note that unzips preserve the ribbon property.

A related operation, *disc unzip*, is unzip done on a capped strand, pushing the tube off in the direction of the framing (in diagram world, in the direction of the blackboard framing), as before. An example is shown in Figure 4; see [WKO2, Section 4.1.3] for details on framings and unzips.

So far all the operations we have introduced had already existed in wTF^o. There is also a new operation is called "puncture", denoted p_e, which diagrammatically simply turns the thick black strand e into a thin red one. The corresponding topological picture is "puncturing a tube", i.e., removing a small disk from it and retracting the rest to its core. Any crossings where e passes under another strand are not affected, while crossings in which e is the over strand turn into virtual crossings.

For simplicity, we place a restriction on which strands can be punctured, namely at each (fully thick black) vertex puctures are only allowed for one of the three meeting strands, as shown on the left of Figure 5. More general puctures could be allowed in a theory with more than one kind of "string to tube" vertex. The right of the same figure shows that when puncturing one of the thick strands of a mixed vertex, the puncture "spreads". Topologically, this is because the mixed vertex represents a string attached to a tube, so when puncturing e, the entire tube retracts to its core. Finally, a capped tube disappears (deformation retracts to a point) when punctured.

In the other direction, consider an arrow diagram on the capped/stringed vertex. One may assume that there are only arrow tails on the capped strand under the vertex: any arrow head may be commuted using $\overrightarrow{STU}$ relations towards the cap, where it is killed by the CP relation[6]. On the thin red strand there are only arrow heads. To construct φ, first "push" the arrow tails (denoted "t") from the capped strand up across the vertex using the VI relation. Since tails vanish on the thin red strand, they simply slide past the vertex. Once the capped side is cleared, continue by sliding the arrow heads "h" up from the thin red string to the strand above the vertex. Now the cap relation kills any arrow heads on the capped strand, so once again they simply slide past the vertex. The result placed on a single thick black strand is shown in Figure 11.

Aside This matches with the theme of [OV].

It is clear that ψ is well-defined, we leave it to the reader to check that so is φ as a short exercise. Given that both maps are well-defined, it is clear that they are inverses of each other. □

2.5. **The homomorphic expansion.** As discussed in [WKO2, Section 2.3], an expansion for $\widetilde{wTF}$ is a map $Z^w : \widetilde{wTF} \to \mathcal{A}^{sw}$ with the property that the associated graded map $\operatorname{gr} Z^w : \mathcal{A}^{sw} \to \mathcal{A}^{sw}$ is the identity map $\mathrm{id}_{\mathcal{A}^{sw}}$. A homomorphic expansion is an expansion which also intertwines each operation of $\widetilde{wTF}$ with its arrow diagrammatic counterpart. In [WKO2, Theorems 4.9 and 4.11] we proved that the existence of solutions for the Kashiwara–Vergne equations implies that there exsists a homomorphic expansion for $\widetilde{wTF}^o$. In fact that homomorphic expansions[7] for $\widetilde{wTF}^o$ are in one-to-one correspondence with solutions to the Kahiwara-Vergne problem.

The point of this paper is to provide a topological construction for such a homomorphic expansion (and hence for a solution of the Kashiwara–Vergne conjecture), and this is easier to do for the slightly more general space $\widetilde{wTF}$.

Let $\mathcal{A}^{osw} \subseteq \mathcal{A}^{sw}$ denote arrow diagrams on $\widetilde{wTF}^o$ skeleta, the associated graded space of $\widetilde{wTF}^o$. One of the key results of [WKO2, Section 4.3] is the characterisation of homomorphic expansions of $\widetilde{wTF}^o$. For any (group-like) homomorphic expansion $Z^{ow} : \widetilde{wTF}^o \to \mathcal{A}^{osw}$, the value $Z^{ow}(\overcrossing)$ is uniquely determined and equals $R = e^{a_{12}}$, where a_{12} denotes a single arrow from the over strand 1 to the under strand 2.

To state the full characterisation, we use co-simplicial notation in subscripts. For example, for $R = e^{a_{12}} =\in \mathcal{A}^{sw}(\uparrow_2)$, $R_{13} = e^{a_{13}}$ and $R_{23} = e^{a_{23}}$ in $\mathcal{A}^{sw}(\uparrow_3)$ are the diagrams where R is placed on strands 1 and 3, and 2 and 3, respectively. $R_{(12)3} \in \mathcal{A}^{sw}(\uparrow_3)$ is obtained by doubling the first strand of R and placing it on strands 1 and 2, and placing the second strand of R on strand 3. Similarly for $V \in \mathcal{A}(\uparrow_2)$, $V_{12} \in \mathcal{A}(\uparrow_3)$ denotes V placed on the first two starnds, et cetera. Thus $R_{(12)3} = e^{a_{13}+a_{23}}$

Fact 2.5. *A filtered, group-like ~~map~~ $Z^{ow}(..)$ $Z^{ow} : \widetilde{wTF}^o \to \mathcal{A}^{osw}$ is a homomorphic expansion if and only if the Z^{ow}-values V and C of the positive vertex and the cap, respectively, satisfy the following equations:* $Z^{ow}(..)$

(1) R4 Equation:

$$V_{12}R_{(12)3} = R_{23}R_{13}V_{12} \quad \text{in } \mathcal{A}^{sw}(\uparrow_3). \tag{R4}$$

[6] This argument also appears in [WKO2], for example as the basic idea for the proof of Fact 2.2.

[7] Subject to the minor technical condition that the value of the vertex doesn't contain isolated arrows.

(2) Unitarity Equation:

$$V \cdot A_1A_2(V) = 1 \quad \textit{in } \mathcal{A}^{sw}(\uparrow_2), \tag{U}$$

where A_1 and A_2 denote the antipode operations.

(3) Cap Equation[8]:

$$C_{(12)}V_{12}^{-1} = C_1C_2 \quad \textit{in } \mathcal{A}^{sw}(\downarrow_2), \tag{C}$$

where the subscripts mean strand placements as in the R4 Equation.

We begin by showing that finding a homomorphic expansion for $\widetilde{wTF}$ is no harder than finding one for wTF^o.

Theorem 2.6. *Homomorphic expansions for wTF^o are in one-to-one correspondence with homomorphic expansions for $\widetilde{wTF}$ via unique extension and restriction.*

Proof. Every element of wTF^o is also in $\widetilde{wTF}$, hence any Z^w restricts to a homomorphic expansion Z^{ow} of wTF^o. Every element of $\widetilde{wTF}$ is the result of puncturing – possibly on multiple strands – an element of wTF^o, and Z^w is required to commute with punctures. Hence any Z^{ow} uniquely extends to a Z^w. □

$$\begin{tikzcd} wTF^o \arrow[r, hook] \arrow[d, "Z^{ow}"] & \widetilde{wTF} \arrow[d, "Z^w"] \\ \mathcal{A}^{osw} \arrow[r, hook] & \mathcal{A}^{sw} \end{tikzcd}$$

In [WKO2, Section 4.4] we showed that short arrows – arrows whose head and tail is not separated by any other arrow endings – supported on either strand of V don't affect whether Z^w is a homomorphic expansion. That is, if Z^w is a homomorphic expansion and a is a linear combination of short arrows, then replacing V by e^aV gives rise to another homomorphic expansion. Hence, in [WKO2] we typically assume there are no short arrows in V, this motivates the following definition:

Definition 2.7. A homomorphic expansion Z is *v-small* if there are no short arrows in the Z-value V of the positive vertex.

As it turns out, the value of the left-punctured vertex is trivial under any v-small homomorphic expansion. This fact will be useful later, so we prove it here.

$Z^w(\ )=1$ **Lemma 2.8.** *For any v-small homomorphic expansion Z^w, ? that is, the Z^w-value of a left punctured vertex is trivial.*

Proof. Recall from [WKO2, Proof of Theorem 4.9] that the Z^w-value V of the positive (not punctured) vertex can be written as $V = e^be^t$, where b is a linear combination of wheels only and t (denoted uD in [WKO2]) is a linear combination of trees. Puncturing the left strand of V kills all arrow diagrams with tails on the left strand. Diagrams that survive are wheels, and trees all of whose tails are on the right side strand. However, if all tails of a tree are supported on one strand, then the tree is a single arrow, due to TC and the anti-symmetry of the trivalent arrow vertices, thus the only surviving trees are simple arrows directed from right to left. Observe that all of these arrow diagrams commute with each other in $\mathcal{A}^{sw}(\uparrow_2)$.

Denote the value of the punctured vertex by $p_1V = e^{p_1(b)}e^{p_1(t)}$. Recall that V must satisfy the Unitarity Equation of Fact 2.5, so $p_1V \cdot A_1A_2(p_1V) = 1$. Since wheels have only tails, $A_1A_2(p_1(b)) = p_1(b)$. Each arrow has one head, so $A_1A_2(p_1(t)) = -p_1(t)$. Hence, using

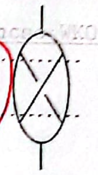
[8]For convenience we state the Cap Equation phrased for caps at the bottom of strands, hence the difference from the equivalent formulation in [WKO2].

commutativity, $p_1V \cdot A_1A_2(p_1V) = e^{2p_1(b)} = 1$, which implies that $p_1(b) = 0$. As for $p_1(t)$, one can show that there are no arrows pointing from the right to the left strand by a direct computation in degree 1. $\square$

3. Proof of Theorem 1.1

3.1. Proof of Part (1). We prove Part 1 in two steps: first verifying the easier "tree level" case, which nonetheless contains the main idea, then in general.

3.1.1. *Tree level proof of Part (1).* Let $\mathcal{A}^{tree}$ denote the quotient of $\mathcal{A}^{sw}$ by all wheels, and let $\pi : \mathcal{A}^{sw} \to \mathcal{A}^{tree}$ denote the quotient map (cf [WKO2, Section 3.2]). Part (1) of the main theorem is the same as stating that Z^w is determined by Z^u. Z^w, in turn is determined by the values V and C of the positive vertex and the cap [WKO2, Sections 4.3 and 4.5], so one only needs to show that V and C are determined by Z^u. Proving this "on the tree level" means showing only that $\pi(V)$ and $\pi(C)$ are determined by Z^u. In particular, observe that since C is a linear combination of products of wheels (Fact 2.2), we have $\pi(C) = 1$, so we only need to show that $\pi(V)$ is determined by Z^u.

Let B^u denote the "buckle" *sKTG*, as shown on the right (ignore the dotted lines for now). All edges are oriented up, and by the drawing conventions of [WKO2, Section 4.6] all the vertices in the bottom half of the picture are negative and $B^u =$  the ones in the top half are positive. Let $B^w = a(B^u) \in \widetilde{w\mathcal{T}F}$, and $\beta^u := Z^u(B^u)$. Note that β^u can be thought of as a chord diagram on four strands: use VI relations to move all chord endings to the "middle" of the skeleton, between the dotted lines on the picture. Hence, we write $\beta^u \in \mathcal{A}^u(\uparrow_4)$. Let $\beta^w = \alpha(\beta^u)$, and note that by the compatibility of Z^u and Z^w we have $\beta^w = Z^w(B^w)$. We will perform a series of operations on B^w and $\pi(\beta^w)$ to recover $\pi(V)$ from it.

First, connect (a circuit algebra operation in $\widetilde{w\mathcal{T}F}$) a positive vertex to the bottom of B^w, as shown in Figure 12. Then unzip the edge marked by u, and puncture the edges marked e and e'. Then attach a cap (once again a circuit algebra operation) to the thick black end at the bottom. Finally, unzip the capped strand.

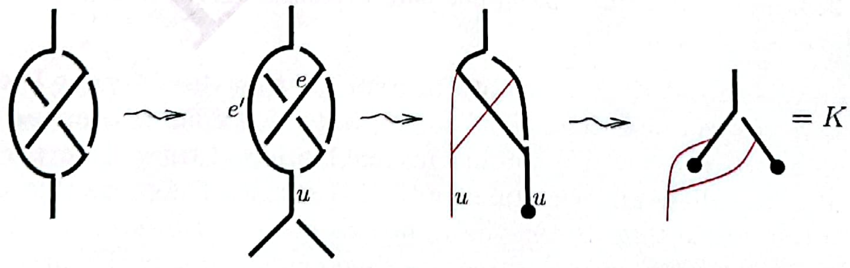


Figure 12. From the "buckle" β^w to the (modified) vertex.

Call the resulting w-foam K, as shown at the right in Figure 12. What is $Z^w(K)$? Due to the homomorphicity of Z, it is obtained from β^w by performing the same series of operations in the associated graded: a circuit algebra composition with V, unzip, punctures, circuit algebra composition with C, and disc unzip. Notice that the left strand of that attached vertex got punctured, and hence by Lemma 2.8 the attached value V cancels.[9] $Z^w(K)$ still depends on the value C. At the tree level, since $\pi(C) = 1$, $\pi(Z^w(K))$ can be computed from β^w by performing punctures and unzips. Since $\beta^w = \alpha(\beta^u)$, this means that $\pi(Z^w(K))$ is determined by Z^u.

On the other hand, note that the space of chord diagrams on the skeleton of K is the space $\mathcal{A}(\uparrow_2)$ by Lemma 2.4 and VI. Note also that K is a circuit algebra combination of a vertex, two left-punctured right-capped vertices and an all-red-strings vertex, and the Z^w-values of the latter three are trivial. So $\pi(Z^w(K)) = \pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. Hence, $\pi(V)$ is determined by Z^u as needed. □

Part1Proof

3.1.2. *Complete proof of Part (1).* In the previous subsection we showed that Z^u determines $\pi(V) \in \mathcal{A}^{tree}(\uparrow_2)$. Now we show that in turn, $\pi(V)$ determines both V and C uniquely, using a perturbative argument.

By contradiction, assume this is not the case, in particular, first assume that there exist $V \neq V'$, both of which are vertex values of Z^u-compatible homomorphic expansions, such that $\pi(V) = \pi(V')$. Let v denote the lowest degree term of $V - V'$. Note that v is primitive and $v \in \ker \pi$, so v is a homogeneous linear combination of wheels. By the Unitarity Equation of Fact 2.5, we have $A_1A_2(v) = -v$. Recall that A_i reverses the direction of the strand i and multiplies each arrow diagram by (-1) to the number of heads on that strand. Since v has only tails, $A_1A_2(v) = v$, so $v = -v$, so $v = 0$, a contradiction. Therefore, $\pi(V)$ determines V uniquely.

Now we show that V determines C uniquely. Assume there are different values C and C' in $\mathcal{A}^{sw}(\downarrow)$ so that (V, C) and (V, C') are both vertex-cap value pairs of Z^u-compatible homomorphic expansions. Let c denote the lowest degree term of $C - C'$, then c is a scalar multiple of a single wheel. The Cap Equation of Fact 2.5 implies $c_{(12)} = c_1 + c_2$ in $\mathcal{A}^{sw}(\downarrow_2)$.

There is a well-defined linear map $\omega : \mathcal{A}^{sw}(\downarrow_2) \to \mathbb{Q}[x, y]$ sending an arrow diagram – which has arrow tails only on each strand – to "x to the power of the number of tails on strand 1, times y to the power of the number of tails on strand 2". Assume $c = \alpha w_r$, where w_r denotes the r-wheel, and $\alpha \in \mathbb{Q}$. Then $0 = \omega(c_{(12)} - c_1 - c_2) = \alpha((x + y)^r - x^r - y^r)$, so either $r = 1$ or $\alpha = 0$. But $w_1 = 0$ in $\mathcal{A}^{sw}$ by the RI relation, hence $\alpha = 0$ and thus $c = 0$, a contradiction. □

AETFormula

3.2. **Proof of Part (2).** In this section we compute V, the value of the vertex, from Φ, the Drinfel'd associator determining Z^b, using the construction of Part (1). In Appendix A we also show that this result translates to the [AET] formula for Kashiwara-Vergne solutions in terms of Drinfel'd associators.

3.2.1. *From Φ to V.* To compute V, consider once again the w-tangled foam K on the right of Figure 12. On one hand, $Z^w(K)$ can be computed directly from the generators: $Z^w(K) = C_1C_2V_{12} \in \mathcal{A}^{sw}(\uparrow_2)$, since the values of the left-punctured vertices are trivial. A

oth hands should e in the same aragraph, or each should have its own paragraph.

On the other hand, one can compute $Z^w(K)$, using the compatibility with Z^u. ~~Recall~~ Note that B^u is the closure B of the parenthesised braid B^b shown in Figure 13, $B^w = a(B^u)$, $\beta^u = Z^u(B^u)$, and $\beta^w = Z^w(B^w)$. By the compatibility of Z^w with Z^u, we have

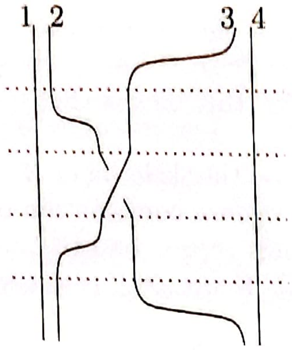
$$\beta^w = Z^w(B^w) = \alpha(Z^u(B^u)) = \alpha(\beta^u).$$

[9]Any short arrows would also cancel when the right strand is capped.

A: Hence if we know $Z^w(K)$, we know V.

B: (in the sense of $\mathbb{Q}$(eq:cl))

$B^b =$



$$
\begin{array}{ll}
\Phi_{(12)34} & \longrightarrow \Phi(a_{23}, a_{43}) \\
\Phi^{-1}_{123} & \longrightarrow 1 \\
R_{32} & \longrightarrow e^{a_{23}/2} \\
\Phi_{132} & \longrightarrow 1 \\
\Phi^{-1}_{(13)24} & \longrightarrow \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})
\end{array}
$$

Figure 13. Computing β^b. Strands are numbered at the top and multiplication is read from bottom to top; the rightmost column lists the images of the factors under $p_1p_2\alpha$.

How does $Z^w(K)$ differ from β^w? To obtain K, a vertex and a cap were attached to B^w, two strands were punctured and the cap unzipped, as in Figure 12. The Z^w-value of the added vertex cancels when its left strand is punctured, however, the value of the cap remains and gets unzipped. Thus, in loose notation, $Z^w(K) = u(C) \cdot p^2(\beta^w)$, where p^2 denotes the two punctures – we will compute this value explicitly in terms of associators shortly.

To equate the two approaches, we need to express this value as an element of $\mathcal{A}^{sw}(\uparrow_2)$, by applying the isomorphism of Lemma 2.4. We thus obtain

$$C_1C_2V_{12} = \varphi(u(C)p^2(\beta^w)). \tag{3}$$

Through a careful analysis of the right hand side, this will imply ~~the~~ formula (6) stated in Teorem Theorem 1.1. In other words, we want to compute

$$\Upsilon := \varphi(u(C)p^2(\beta^w)).$$

To achieve this, we use that $\beta^w = \alpha(\beta^u)$, and compute β^u in terms of the Drinfel'd associator Φ associated to Z^u. By the compatibility of Z^u and Z^b, it is enough to compute $\beta^b := Z^b(B^b)$. The result can be read from the picture in Figure 13:

$$\beta^b = \Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34}.$$

Do we need to repeat the "cosimplicial notation"? [To interpret this formula, recall that the associator Φ is an element of $\mathcal{A}^{hor}(\uparrow_3)$, and the cosimplicial notation used in the subscripts show which strands diagrams are placed on. For example, the notation $\Phi^{-1}_{(13)24}$ means doubling the first strand of Φ^{-1} and placing the resulting chord endings on strands 1 and 3, as well as placing the chord endings from the other two strands of Φ^{-1} on strands 2 and 4.] Also recall that $R = e^{c/2}$, where c is a single horizontal chord between two strands (and R_{32} means that this chord runs between strands 3 and 2).

As β^u is the tree closure of β^b, it is given by the same formula interpreted as an element of $\mathcal{A}^u(\uparrow_4)$. One then applies α to obtain $\beta^w = \alpha(\beta^u)$. After the vertex and cap attachment, of Figure 12, strands 1 and 3 are punctured and strands 2 and 4 are capped, and in this strand numbering, $u(C) = C_{24}$ Therefore, we have

$$\Upsilon = \varphi\left(C_{24} \cdot p_1p_3\alpha(\Phi^{-1}_{(13)24}\Phi_{132}R_{32}\Phi^{-1}_{123}\Phi_{(12)34})\right).$$

Next, we analyse how the puctures and α act on factors of β^b. First observe that $p_3\alpha(R_{32}) = e^{a_{23}/2}$, where a_{ij} is a single arrow pointing from strand i to strand j.

Perhaps we should assemble some "$\alpha\Phi$ facts" lemma?

Figure 14. Strand numbering convention for K and V.

To compute $p_1p_3(\alpha\Phi^{-1}_{123})$, recall that Φ_{123} can be expressed as a power series in non-commuting variables c_{12} and c_{23} (i.e., chords between strands 1—2 and 2—3, respectively): $\Phi_{123} = \Phi(c_{13}, c_{23})$. Given that $\alpha(c_{12}) = a_{12} + a_{23}$, we have $p_1(\alpha(c_{12})) = a_{21}$. Similarly, $p_3(\alpha(c_{23})) = a_{23}$, therefore $p_1p_3(\alpha\Phi^{-1}_{123}) = \Phi^{-1}(a_{21}, a_{23})$. By the TC relation, a_{21} and a_{23} commute, and a basic fact about Drinfel'd associators is that image of Φ in a quotient where its variables commute is 1. Thus, $p_1p_3(\alpha\Phi^{-1}_{123}) = 1$. Similarly, $p_1p_3(\alpha\Phi_{132}) = 1$ because $p_1p_3\alpha(c_{13}) = 0$.

Since strands 1 and 3 are both punctured, no arrows can be supported between these two strands, hence $p_1p_3\alpha(\Phi_{(12)34}) = p_3\alpha(\Phi_{234})$. Writing Φ as a power series, $\Phi_{234} = \Phi(c_{23}, c_{34})$, and $p_3\alpha(\Phi(c_{23}, c_{34})) = \Phi(a_{23}, a_{43})$.

Similarly, $\Phi^{-1}_{(13)24} = \Phi^{-1}(c_{(13)2}, c_{24})$, where $c_{(13)2} = c_{12} + c_{32}$. The cyclic symmetry property of associators implies $\Phi(c_{ij}, c_{jk}) = \Phi(c_{ij}, -c_{ij} - c_{ik})$. Hence, $\Phi^{-1}(c_{(13)2}, c_{24}) = \Phi^{-1}(c_{(13)2}, -c_{(13)2} - c_{(13)4})$, so $p_1p_3\alpha\Phi^{-1}_{(13)24} = \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)})$. To summarise,

$$\Upsilon = \varphi\left(C_{24} \cdot \Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right).$$

Note that the expression $\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})$ has only arrow tails on strands 2 and 4, and therefore commutes with C_{24} by the TC relation. Hence, by the definition of φ,

$$\begin{aligned}\Upsilon &= \varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43}) \cdot C_{24}\right) \\ &= \varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right) \cdot \varphi(C_{24}).\end{aligned}$$

applying

Furthermore,since arrow endings from strand 2 are "pushed" to strand 1 when ~~performing~~ φ, and arrow endings from strand 2 are pushed to strand 2, we have $\varphi(C_{24}) = C_{12}$. Terefore,

$$V_{12} = C_1^{-1}C_2^{-1}\Upsilon = C_1^{-1}C_2^{-1}\varphi\left(\Phi^{-1}(a_{2(13)}, -a_{2(13)} - a_{4(13)}) \cdot e^{a_{23}/2} \cdot \Phi(a_{23}, a_{43})\right) C_{12},$$

as stated in part (2) of Theorem thm:main 1.1.

of [AET, ...]

Matching this result to the Alekseev–Enriquez–Torossian formula is technical, and not used anywhere else in the paper, hence we defer this to Appendix app:AET A.

subsec:DT

3.3. Proof of part (3): the double tree construction.

It remains to prove that the values of V and C, which we proved in Section subsec:PartIProof 3.1.2 are determined by Z^u, indeed give rise to a homomorphic expansion of $\widetilde{wTF}$. In other words, one needs to show that they satisfy the three equations of Fact fact:EquationsForZ 2.5. Unfortunately, doing this directly seems difficult.

Note that eq:R4 (R4), which is in some sense the "main equation", is an equality between different planar algebra compositions of generators. Hence, the proof would be much easier if Z^u were to be a planar algebra map. This unfortunately makes no sense, as *sKTG* is not a planar algebra but a different, more complicated algebraic structure. The reader might ask, why work with a space as inconvenient as *sKTG* instead of, say, the planar algebra of trivalent tangles? The answer is that the existence of a homomorphic expansion is a highly non-trivial property, and in particular ordinary trivalent tangles do not have one. Even without trivalent vertices, ordinary tangles, or u-tangles, do not have have a homomorphic

$$\begin{tikzcd}
T \arrow[r, mapsto] & \wp(T) \arrow[r, mapsto] & \wp(T) \arrow[r, mapsto] & \kappa\wp(T) \arrow[r, mapsto] & \hat{T} \\
\rotatebox{90}{$\Subset$} & \rotatebox{90}{$\Subset$} & \rotatebox{90}{$\Subset$} & \rotatebox{90}{$\Subset$} & \rotatebox{90}{$\Subset$} \\
u\mathcal{T}T \arrow[r, "\wp"] \arrow[dr, phantom, "(4)"] \arrow[ddrrrr, bend right, "\xi"'] & sKTG \arrow[r, "a"] \arrow[d, "Z^u"] \arrow[dr, phantom, "(1)"] & \widetilde{u\mathcal{T}F} \arrow[r, "\kappa"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(2)"] & \widetilde{u\mathcal{T}F} \arrow[r, "pou"] \arrow[d, "Z^w"] \arrow[dr, phantom, "(3)"] & \widetilde{u\mathcal{T}F} \arrow[d, "Z^w"] \\
 & \mathcal{A}^u(\wp(T)) \arrow[r, "\alpha"] & \mathcal{A}^{sw}(\wp(T)) \arrow[r, "\kappa\circ\hat{C}"] \arrow[d, phantom, "(4)"] & \mathcal{A}^{sw}(\kappa\wp(T)) \arrow[r, "pou"] & \mathcal{A}^{sw}(\hat{T}) \arrow[d, "\varphi"] \\
 & & \; & & \mathcal{A}^{sw}(T)
\end{tikzcd}$$

Figure 20. Comparing ξ and Z^w, assuming that Z^w exists. fig:BigCom

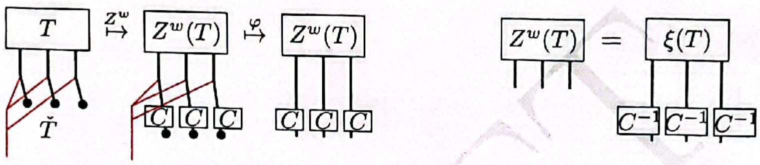


Figure 21. Computing $Z^w(\check{T})$ and $Z^w(T)$. fig:ZTChec

tree, the strand directly underneath the twist is capped and uzipped, and hence the value of the twist cancels by the CP relation.

Now observe that the right side picture of Figure 19 only differs from $\wp(T)$ in the choices of binary trees, which do not change the value of ξ by Lemma 3.1. $\square$

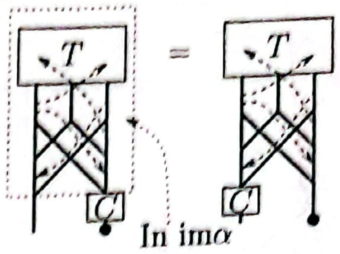
The following lemma clarifies the relationship between the map ξ and the homomorphic expansion Z^w that we're aiming to construct:

patibility **Lemma 3.3.** *If there exists a homomorphic expansion Z^w for $\widetilde{u\mathcal{T}F}$ compatible with Z^u, and $T \in u\mathcal{T}T$ is a tangle with n ends, then $Z^w(a(T)) = \xi(T)\cdot(C^{-1})^n$, where $C = Z(\downarrow)$, and $\xi(T)$ is multiplied by C^{-1} at each tangle end of T, as in Figure 21.*

Proof. Assume there exists a homomorphic expansion Z^w compatible with Z^u. We use, as in Figure 20, the homomorphicity of Z^w and its compatibility with Z^u to show that $\xi(T) = Z^w(\check{T})$, where $\check{T}$ is as in Equation 4 and shown in Figure 21 on the left.

If the diagram in Figure 20 commutes, then for any $T \in u\mathcal{T}T$ and any Z^u-compatible Z^w, we have $\varphi(Z^w(\check{T})) = \xi(T)$. Since Z^w is a circuit algebra homomorphism, $Z^w(\check{T})$ can be obtained from $Z^w(T)$ by attaching the Z^w-value of a left-punctured right-capped vertex at each tangle end, as illustrated in Figure 21. By Lemma 2.8 we have $Z^w(\curlywedge) = 1$, so the only additions are C values at each capped end, as shown in Figure 21. This can then be interpreted as a value in $\mathcal{A}^{sw}(T)$ via the isomorphism φ of Lemma 2.4. This implies the statement of the Lemma.

It's ~~All that's~~ left to show is the commutativity of the diagram in Figure 20. The square (1) is the assumed compatibility of Z^u and Z^w. In square (2), recall the map κ denotes the circuit algebra operation of attaching a cap at the bottom right end of the w-foam. The map $\hat{C}$ denotes the circuit algebra operation which attaches a value $C = Z(\downarrow)$ at the end of the strand. Thus, the commutativity of square (2) is implied by the homomorphicity of Z^w with respect to circuit algebra composition (as a binary operation). The square (3) is commutative due to the homomorphicity of Z^w with respect to punctures and disc unzips.



The commutativity of the heptagon (4) would be true by definition, if not for the map $\hat{C}$ (multiplication by the cap value). We show that, in fact, the value C cancels after punctures, by a property of arrow diagrams in the image of α, called *tail-invariance*, shown in Figure 17 (see [WKO2], Remark 3.14 and early in Section 3.3). In the current situation tail invariance means that the value C, which has only arrow tails, can be moved from one tangle end to the other, as shown on the right. Consequently, C cancels when the left strand is punctured. □

Remark 3.4. In Lemma 3.3 we assume by convention that all tangle ends of T are oriented upwards (towards T). If k tangle ends are oriented down, the corresponding cap values appear with their orientations switched: $Z^w(aT) = \xi(T)\cdot(C^{-1})^{n-k}(S(C)^{-1})^k$.

Corollary 3.5. *If there exists a homomorphic expansion Z^w for $\widetilde{w\mathcal{TF}}$ compatible with Z^u, then $\pi(V) = \pi(\xi(\curlywedge))$, where V is the Z^w-value of the vertex, and π is the tree projection. This uniquely determines Z^w.*

Proof. The first statement is an immediate consequence of Lemma 3.3. The second was shown in Section 3.1.2. □

Thus, the map ξ uniquely determines Z^w, assuming that Z^w exists, and we have shown how to explicitly compute $\pi(V)$ from Z^b through ξ. What remains to be proven is that:

A

(1) Z^w, as constructed from ξ, is compatible with Z^u: see Proposition 3.8. below.
(2) The restriction of Z^w to $a(u\mathcal{TT})$ is a planar algebra map (see Theorem 3.11), below and thus Z^w satisfies the (R4) equation.
(3) Z^w satisfies the (U) and (C) equations, ~~hence it is~~ a homomorphic expansion of $\widetilde{w\mathcal{TF}}$ compatible with Z^u: see Theorem 3.16. below
(4) The versions of Z^w obtained from the buckle construction of Section 3.1.1 and the double tree construction (i.e., the map ξ) coincide: see Lemma 3.14. below

3.3.2. *Z^w is a homomorphic expansion.* The goal for this subsection is to ~~carry out the proof~~ complete the sketch outlined above. We begin by ~~proving~~ showing that Z^w_ξ, ~~as given by ξ,~~ is compatible with Z^u. This requires a technical lemma, in which we compute the ξ-value of a vertical strand:

Lemma 3.6. *For a single un-knotted strand, $\xi(\uparrow) = \alpha(\nu^{1/2})$, where $\nu \in \mathcal{A}^u(\uparrow)$ denotes the Kontsevich integral of the un-knot*[13]. perhaps.

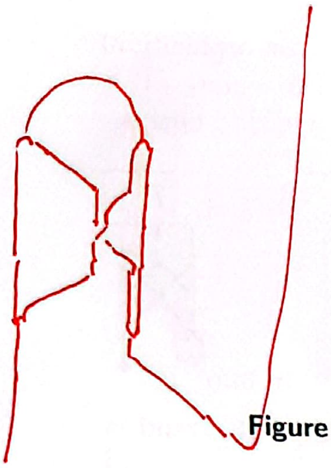
Proof. We apply φ to $\uparrow$, as shown in Figure 22, and comupte $Z^u(\varphi(\uparrow))$ using the finite generation property of *sKTG* and the homomorphicity of Z^u. In [WKO2, Section 5.2] we gave an algorithm for writing any *sKTG* as an *sKTG*-composition of generators (the primary operation in *sKTG* is *tangle insertion*, see [WKO2, Figure 22]). Feeding $\varphi(\uparrow)$ into this algorithm, one needs to "curve up" one strand as in Figure 22, in this case the strand on the right (the choice of strand doesn't affect the outcome).

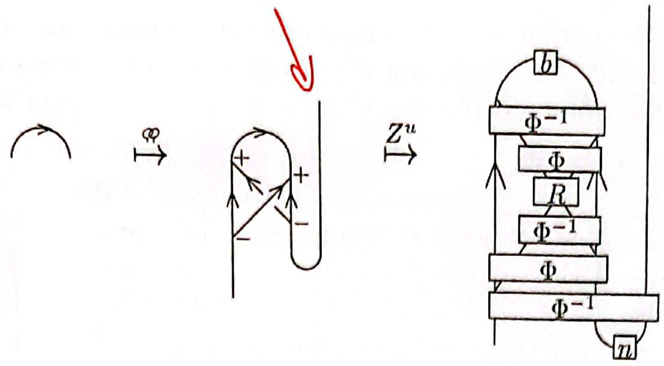
[13]The value of ν was conjectured in [BGRT] and proven in [BLT]. Note that ν involves wheels only.

A: Can we call it Z^w_ξ? As it is now, we have two Z^w's: The "assumed" one and the "constructed" one. Better keep them distinct until the construction is finished.

I'm missing

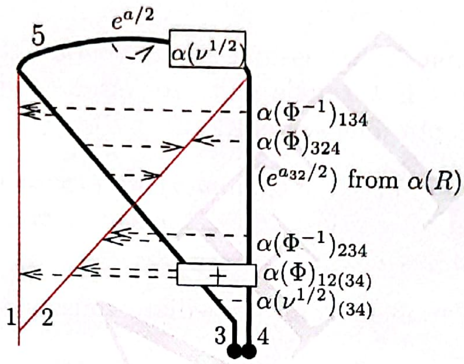
I'm confused — why the up-line?





Perhaps we shall assemble some "KZ Facts" lemma?

Figure 22. The double tree map composed with Z^u, applied to a single strand.



There should be a better separation between the ingredients, and a clearer indication of which one is managed in which paragraph.

Figure 23. The value of $pu\kappa\alpha Z^u\varphi(\uparrow)$.

The chord diagram $Z^u(\varphi(\uparrow))$ is shown in Figure 22, expressed in terms of the generators of *sKTG* described in [WKO2, Proposition 4.13]:

- the value Φ of the (right) *associator graph*: Φ is a (horizontal-chord) Drinfel'd associator
- the value R of the (right) *twist graph*: $R = e^{c/2}$, where c is a single chord, and
- the values n and b of the *noose* and *balloon* graphs, respectively.

shorten to one sentence

In $\xi(\uparrow)$, Z^u is followed by α, a cap attachment, unzips and punctures. As explained in [WKO2, Section 4.6], there is possibly a one-parameter freedom in the values of n and b, but we know that and $\alpha(b) = e^{a/2}\alpha(\nu)^{1/2}$, and $\alpha(n) = e^{-a/2}\alpha(\nu)^{1/2}$. Note that the exponential part of n cancels by the CP relation once the cap is attached. The value $pu\kappa\alpha Z^u\varphi(\uparrow) \in \mathcal{A}^{sw}$ is shown in Figure 23 and explained below.

Recall that α maps a chord to the sum of its two possible orientations. However, when one supporting strand is punctured, only one of these orientations survive. Hence, for example, $p_2(\alpha(R_{23})) = (e^{a_{32}/2})$. Figure 23 shows a schematic picture of $pu\kappa\alpha Z^u\varphi(\uparrow)$ with exponentials and associators indicated by single arrows. To explain the notation for associators, recall that $\Phi \in \mathcal{A}^{hor}(\uparrow_3)$ can be written as a power series in any two of the three generators of $\mathcal{A}^{hor}(\uparrow_3)$: c_{12}, c_{23} and c_{13}. For example, $\Phi(c_{12}, c_{23}) = \Phi(c_{12}, -c_{12} - c_{23})$. For each associator above, we chose the presentation in which $pu\kappa\alpha(\Phi)$ is of the simplest form, as follows.

Wasn't this explained already?

The top associator of Figure 22, after applying a VI relation, is written as $\Phi^{-1}_{13(24)}$ in the strand numbering of Figure 23. We write this in terms of c_{13} and $c_{1(24)} = c_{12} + c_{14}$, since after the puctures $p_1\alpha(c_{13}) = a_{31}$ and $p_1p_2\alpha(c_{1(24)}) = a_{41}$, thus

$$p_1p_2\alpha\Phi^{-1}(c_{13}, -c_{13} - c_{1(24)}) = \Phi^{-1}(a_{31}, -a_{31} - a_{41}).$$

fig:dtstrand2

This is reflected in Figure 23 in drawing only the a_{31} and a_{41} arrows for this associator. Notice that the tail of a_{41} can be "pulled over the top along strand 5" using the VI relations and the fact that $e^{a/2}\alpha(\nu)$ is a local arrow diagram on a single strand, hence it is central. Thus, $a_{41} = a_{31}$, making the second argument of Φ vanish, and therefore

$$p_1p_2\alpha(\Phi^{-1}_{13(24)}) = 1.$$

Second from top we have

$$p_1p_2\alpha(\Phi_{324}) = p_1p_2\alpha(\Phi(c_{23}, c_{24})) = \Phi(a_{32}, a_{42}).$$

Applying the same "pull over the top" trick to a_{42}, we obtain $a_{32} = a_{42}$, and since two arguments of Φ commute, we have

$$p_1p_2\alpha(\Phi_{324}) = 1.$$

For the exponential we have,

$$p_1p_2\alpha(R_{23}) = p_1p_2\alpha(e^{c_{23}/2}) = e^{a_{32}/2}.$$

Next,

$$p_1p_2\alpha(\Phi^{-1}_{234}) = p_1p_2\alpha(\Phi^{-1}(c_{23}, -c_{23} - c_{24})) = \Phi(a_{32}, -a_{32} - a_{42}).$$

Once again, this associator cancels by the "pull over the top" trick, noting that the arrow tail also commutes with the arrow tails of the exponential.

Observe that

$$p_1p_2\alpha(\Phi)_{12(34)} = p_1p_2\alpha(\Phi)(-c_{1(34)} - c_{2(34)}, c_{2(34)}) = \Phi(-a_{(34)1} - a_{(34)2}, a_{(43)2}) = 1,$$

as the two arguments of Φ commute by the TC relation, as strands 3 and 4 support only tails.

Isn't it also killed by TC?

WHAT IS THE SIMPLEST ARGUMENT THAT THE BOTTOM WEIRD ASSOCIATOR DIES?

It's not shown in Fig 23!

Next we show that $\alpha(\nu^{1/2})_{(34)}$, which remains from n, cancels as well. Since ν is an exponential of wheels, so is $\alpha(\nu^{1/2}) \in \mathcal{A}^{sw}(\uparrow)$. Recall from Bar-NatanDancso:WKO1 [WKO1, Section 3.8] that wheels in $\mathcal{A}^{sw}$ have two possible orientations. For odd wheels these are negatives of each other by the AS relation, for even wheels they are equal. Hence, α kills odd wheels and multiplies even wheels by 2, as well as orienting them. Let us write $\alpha(\nu^{1/2})$ as $e^{w(x)}$, where $w(x)$ is an (even) power series in x with constant term 0: interpret each monomial x^k as a k-wheel. Then $u(\alpha(\nu^{1/2})) = \alpha(\nu^{1/2})_{(34)} = e^{w(x_3+x_4)}$, where monomials are read as cyclic words and interpreted as wheels on strands 3 and 4. Now slide this arrow diagram up on strands 3 and 4 to strand 5 by VI. Since $\alpha(R)$ has only tails on strand 3, there is no obstruction to doing this. Tails on the punctured strand 1 are zero (TF relation), so each tail on strand 3 slides onto strand 5, whose orientation is compatible with strand 3. In other words we replace x_3 by x_5 in the expression $e^{w(x_3+x_4)}$. On the other side, tails again slide onto strand 5 but now the orientations are opposite, and hence x_4 is replaced by $-x_5$. Thus,

$$p_1p_2\alpha(\nu^{1/2})_{(34)} = e^{w(x_5-x_5)} = 1.$$

Finally, move the top exponential $e^{a/2}$ to strands 3 and 2, using the VI relation at both vertices. The tail of each arrow moves freely from strand 5 to strand 3. The heads commute

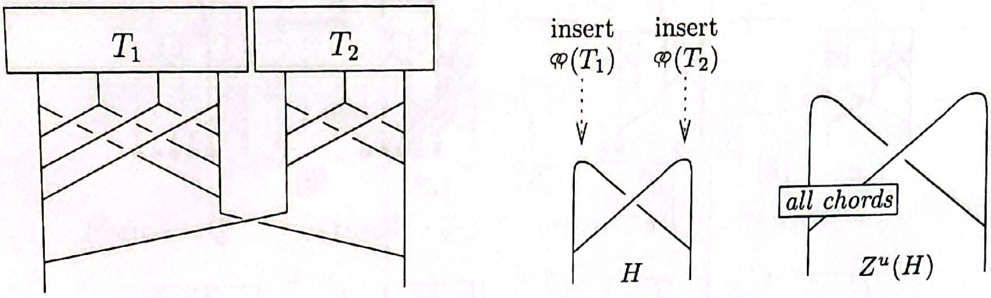


Figure 27. The double tree map applied to a disjoint union of *uTT*-s is the same as inserting the double tree of each individual *uTT* into the *sKTG* H. In $Z^u(H)$ all chords can be pushed into the rectangle shown, using VI relations when necessary. fig:DisjUn

$$\begin{tikzcd}
u\mathit{TT} \arrow[r, "\phi"] \arrow[d, "c"] \arrow[dr, phantom, "(1)"] & \mathit{sKTG} \arrow[r, "Z^u"] \arrow[d, "d^3"] \arrow[dr, phantom, "(2)"] & \mathcal{A}^u \arrow[r, "\alpha"] \arrow[d, "(\cdot e^{-\frac{c}{4}}\nu^{\frac{1}{2}})\circ d^3"] \arrow[dr, phantom, "(3)"] & \mathcal{A}^{sw} \arrow[r, "\kappa"] \arrow[d, "(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))\circ d^3"] \arrow[dr, phantom, "(4)"] & \mathcal{A}^{sw} \arrow[r, "u^{n-1}"] \arrow[d, "(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))\circ d^3"] \arrow[dr, phantom, "(5)"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] \arrow[d, "(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))\circ d^3"] \arrow[dr, phantom, "(6)"] & {} \\
u\mathit{TT} \arrow[r, "\phi"] & \mathit{sKTG} \arrow[r, "Z^u"] & \mathcal{A}^u \arrow[r, "\alpha"] & \mathcal{A}^{sw} \arrow[r, "\kappa"] & \mathcal{A}^{sw} \arrow[r, "u^{n-3}"] & \mathcal{A}^{sw} \arrow[r, "p^{n-2}"] & {}
\end{tikzcd}$$

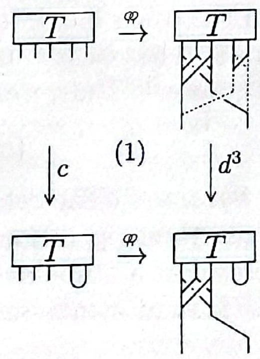
diagram continues...

$$\begin{tikzcd}
\mathcal{A}^{sw} \arrow[r, "p^2"] \arrow[d, "(\cdot e^{-\frac{a}{2}}\alpha(\nu^{\frac{1}{2}}))\circ d^3"] \arrow[drr, phantom, "(7)"] & \mathcal{A}^{sw} \arrow[r, "\varphi"] & \mathcal{A}^{sw} \arrow[r, "\cdot C^{-(n-1)}SC^{-1}"] \arrow[d, "(\cdot\alpha(\nu^{-\frac{1}{2}}))\circ c"] \arrow[dr, phantom, "(8)"] & \mathcal{A}^{sw} \arrow[d, "c"] \\
\mathcal{A}^{sw} \arrow[rr, "\varphi"] & & \mathcal{A}^{sw} \arrow[r, "\cdot C^{-(n-2)}"] & \mathcal{A}^{sw}
\end{tikzcd}$$

Figure 28. Summary of the proof that Z^w commutes with contractions: Z^w is the composition along the entire top and entire bottom horizontal edge of the diagram. fig:Contra

of the n ends of T. Hence we will drop the subscript from c_i and denote this operation simply by c.

We need to show that $Z^w(cT) = cZ^w(T)$, for any $T \in u\mathit{TT}$. Since Z^w is given by the composition of many maps, so this can be restated as the commutativity of the perimeter of a large diagram (shown in Figure 28), which in turn can be broken down to its smaller parts. Throughout this proof, let $T \in u\mathit{TT}$ denote an arbitrary trivalent tangle.



Square (1). This square plays out in *uTT* and *sKTG*, and commutes by inspection, as shown on the right. The three strands to be deleted are indicated by broken lines. Therefore, $d^3\phi(T) = \phi c(T)$.

Square (2). Square (2) is shown schematically below on the left: for the Z^u-values skeleta are indicated but chords are not shown. To prove that square (2) commutes, we use the properties of Z^u with respect to deleting edges in *sKTG*, as stated in Fact 3.9 and Figure 25.